

# \* Fourier Series

\* ای دالة يمكن تمثيلها بمجموع Sin, Cos

Precewise continuous ← دالة تكون

Bounded ←

Periodic ←

$$* f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

→ if  $P(\text{period}) = 2\pi$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

→ if  $P(\text{period}) = 2L$

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

→ even function \* odd function

$$f(-x) = f(x) \quad \left\{ \begin{array}{l} f(-x) = -f(x) \end{array} \right.$$

$$\int_{-L}^L \text{even} = 2 \int_0^L \text{even} \quad \left\{ \begin{array}{l} \int_{-L}^L \text{odd} = 0 \end{array} \right.$$



S-2-1

J-L-L

تأمل،  
(تأمل)

الا سجع

sec 2

\* Rules

$$\text{even} * \text{even} = \text{even}$$

$$\text{even} * \text{odd} = \text{odd}$$

$$\text{odd} * \text{odd} = \text{even}$$

$$\sin A \cos B = \frac{1}{2} [\sin(A-B) + \sin(A+B)]$$

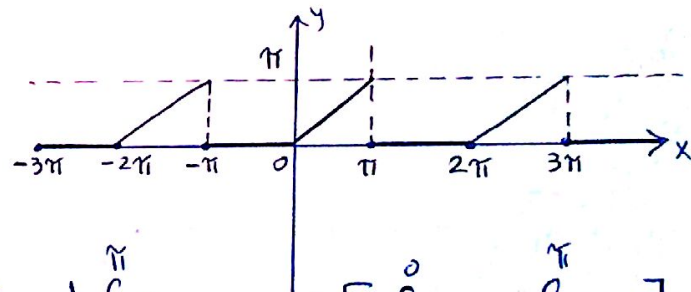
$$\cos A \cos B = \frac{1}{2} [\cos(A-B) + \cos(A+B)]$$

$$\sin A \sin B = \frac{1}{2} [\cos(A-B) - \cos(A+B)]$$

\* Sheet 1 :-

→ find Fourier series for -

$$\textcircled{1} f(x) = \begin{cases} 0 & -\pi \leq x < 0 \\ x & 0 \leq x \leq \pi \end{cases} \quad P=2\pi$$



$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \left[ \int_{-\pi}^0 0 dx + \int_0^{\pi} x dx \right]$$

$$= \frac{1}{\pi} * \frac{x^2}{2} \Big|_0^{\pi} = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi} \left[ \int_{-\pi}^0 0 \cdot \cos nx dx + \int_0^{\pi} x \cos nx dx \right]$$

$$= \frac{1}{\pi} \left( \frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right) \Big|_0^{\pi}$$

$$= \frac{1}{\pi} \left( \frac{\cos n\pi}{n^2} - \frac{1}{n^2} \right)$$

$$= \frac{1}{\pi} \left( \frac{(-1)^n - 1}{n^2} \right) = \frac{(-1)^n - 1}{\pi n^2}$$

$$* \cos(n\pi) = (-1)^n, \sin(n\pi) = 0$$

$$* \cos(2n\pi) = 1, \cos(2n-1)\pi = -1$$



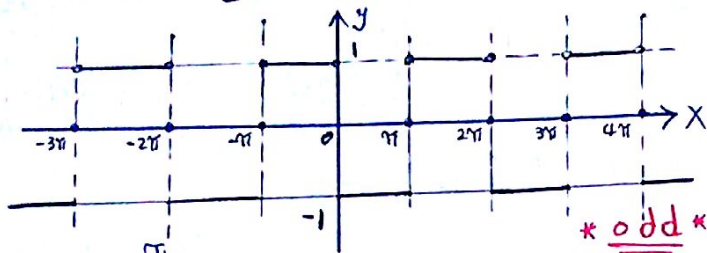
$$b_n = \frac{1}{\pi} \int_0^{\pi} x \sin nx \, dx$$

$$= \frac{1}{\pi} \left( -\frac{x \cos nx}{n} + \frac{\sin nx}{n^2} \right) \Big|_0^{\pi}$$

$$= \frac{1}{\pi} \left( \frac{-\pi(-1)^n}{n} + 0 + 0 \right) = \frac{(-1)^{n+1}}{n}$$

$$\Rightarrow f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left( \frac{(-1)^n - 1}{\pi n^2} \cos nx + \frac{(-1)^{n+1}}{n} \sin nx \right)$$

$$(2) f(x) = \begin{cases} 1 & -\pi < x < 0 \\ -1 & 0 \leq x < \pi \end{cases}$$



$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos nx \, dx = 0$$

odd \* even = odd

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin nx \, dx$$

odd \* odd = even

$$= \frac{2}{\pi} \int_0^{\pi} -\sin nx \, dx = \frac{2}{\pi} \left( \frac{\cos nx}{n} \right) \Big|_0^{\pi}$$

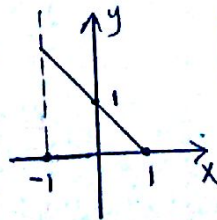
$$= \frac{2}{\pi n} ((-1)^n - 1)$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} \left( \frac{2}{\pi n} ((-1)^n - 1) \sin nx \right)$$

$$(3) f(x) = 1-x, \quad -1 \leq x \leq 1 \quad (P=2)$$

$$a_0 = \frac{1}{2} \int_{-1}^1 (1-x) \, dx$$

$$= \left( x - \frac{1}{2}x^2 \right) \Big|_{-1}^1 = 2$$



$$a_n = \frac{1}{2} \int_{-1}^1 (1-x) \cos n\pi x \, dx$$

$$= \int_{-1}^1 \cos n\pi x \, dx - \int_{-1}^1 x \cos n\pi x \, dx$$

$$a_n = \frac{-\sin(n\pi x)}{n\pi} \Big|_{-1}^1 - \left( \frac{x \sin n\pi x}{n\pi} + \frac{\cos n\pi x}{n^2 \pi^2} \right) \Big|_{-1}^1$$

$$= 0 - \left( \frac{1 \cdot 0}{n\pi} + \frac{\cos n\pi}{n^2 \pi^2} - \left( \frac{-1 \cdot 0}{n\pi} + \frac{\cos(-n\pi)}{n^2 \pi^2} \right) \right)$$

$$= -\frac{2 \cos n\pi}{n^2 \pi^2} = 0$$

$$b_n = \frac{1}{2} \int_{-1}^1 (1-x) \sin n\pi x \, dx$$

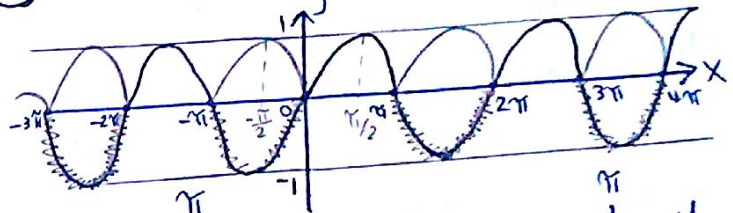
$$= \int_{-1}^1 \sin n\pi x \, dx - \int_{-1}^1 x \sin n\pi x \, dx$$

$$= -\frac{\cos n\pi x}{n\pi} \Big|_{-1}^1 - \left( -\frac{x \cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2 \pi^2} \right) \Big|_{-1}^1$$

$$= \frac{2 \cos n\pi}{n\pi} = \frac{2(-1)^n}{n\pi}$$

$$\Rightarrow f(x) = 1 + \sum_{n=1}^{\infty} \left( \frac{2(-1)^n}{n^2 \pi^2} \sin n\pi x \right)$$

$$(4) f(x) = \sin x, \quad 0 \leq x \leq \pi, \quad P=\pi$$



$$a_0 = \frac{1}{\pi/2} \int_0^{\pi} \sin x \, dx = \frac{2}{\pi} \left( -\cos x \right) \Big|_0^{\pi} = \frac{4}{\pi}$$

hint: لا يجوز حذف الكامل الصور  $\int_{-\pi/2}^{\pi/2}$  من فترة الدالة من  $-\pi/2$  إلى  $\pi/2$  لأننا نحتاج إلى الدالة وليس كلاً (أي نحتاج إلى الدالة وليس كلاً)

$$a_n = \frac{2}{\pi} \int_0^{\pi} \sin x \cos 2nx \, dx \quad \left( \frac{n\pi x}{L} = \frac{n\pi x}{\pi/2} = 2nx \right)$$

$$= \frac{2}{\pi} \int_0^{\pi} \frac{1}{2} \sin(1-2n)x + \sin(1+2n)x \, dx$$

$$= \frac{1}{\pi} \left( -\frac{\cos(1-2n)x}{(1-2n)} - \frac{\cos(1+2n)x}{(1+2n)} \right) \Big|_0^{\pi}$$

$$= -\frac{1}{\pi} \left( \frac{2}{1-2n} + \frac{2}{1+2n} \right)$$

$$= -\frac{1}{\pi} \left( \frac{2+4n+2-4n}{1-4n^2} \right)$$

$$= -\frac{1}{\pi} \left( \frac{4}{1-4n^2} \right) = \frac{4}{\pi(1-4n^2)}$$



$$b_n = \frac{2}{\pi} \int_0^{\pi} \sin x \cdot \sin 2nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \frac{1}{2} (\cos(1-2n)x - \cos(1+2n)x) \, dx$$

$$= \frac{1}{\pi} \left[ \frac{\sin(1-2n)x}{1-2n} - \frac{\sin(1+2n)x}{1+2n} \right]_0^{\pi}$$

$$= 0$$

$$\therefore f(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \left[ \frac{4}{\pi(1-4n^2)} \cos n\pi x \right]$$

⑤ expand  $f(x) = x^2$ ,  $0 \leq x \leq 1$   
 $\rightarrow$  in case of: half-period ( $L=1$ )

- 1) Sine Series only.
- 2) Cosine Series only  $\rightarrow$  (H.W)

$\rightarrow$  Sine Series  $\leftarrow$

$\rightarrow$  it should be odd.

$$a_0 = a_n = 0$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} \, dx$$

odd \* odd = even

$$= 2 \int_0^1 x^2 \sin n\pi x \, dx$$

$$= 2 \left[ -x^2 \frac{\cos n\pi x}{n\pi} + 2x \frac{\sin n\pi x}{n^2 \pi^2} + \frac{2 \cos n\pi x}{n^3 \pi^3} \right]_0^1$$

$$= 2 \left[ \frac{-(-1)^n}{n\pi} + \frac{2(-1)^n}{n^3 \pi^3} - \left[ \frac{2}{n^3 \pi^3} \right] \right]$$

$$= 2 \left( \frac{(-1)^{n+1}}{n\pi} + \frac{2[(-1)^n - 1]}{n^3 \pi^3} \right)$$

$$= \frac{2(-1)^{n+1}}{n\pi} + \frac{4((-1)^n - 1)}{n^3 \pi^3}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \left[ \left( \frac{2(-1)^{n+1}}{n\pi} + \frac{4((-1)^n - 1)}{n^3 \pi^3} \right) \sin n\pi x \right]$$

S-2-2

$\rightarrow$  Cosine Series  $\leftarrow$

$\rightarrow$  it should be even.

$$b_n = 0$$

$$a_0 = \frac{2}{1} \int_0^1 x^2 \, dx$$

$$= 2 \left[ \frac{x^3}{3} \right]_0^1 = \frac{2}{3}$$

$$a_n = \frac{1}{1} \int_{-1}^1 x^2 \cos n\pi x \, dx$$

$$= 2 \int_0^1 x^2 \cos n\pi x \, dx$$

$$= 2 \left[ x^2 \frac{\sin n\pi x}{n\pi} + 2x \frac{\cos n\pi x}{n^2 \pi^2} + 2 \frac{\sin n\pi x}{n^3 \pi^3} \right]_0^1$$

$$= 2 \left( 2x \frac{\cos n\pi x}{n^2 \pi^2} \right) \Big|_0^1$$

$$= 2 * 2 \frac{(-1)^n}{n^2 \pi^2} = \frac{4(-1)^n}{n^2 \pi^2}$$

$$\therefore f(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \left( \frac{4(-1)^n}{n^2 \pi^2} \cos n\pi x \right)$$

